Western Australian
Junior Mathematics Olympiad 2005

Individual Questions **100 minutes**

General instructions: *Each solution in this part is a positive integer less than 100. No working is needed for Questions 1 to 9. Calculators are* ***not*** *permitted. Write your answers on the answer sheet provided.*

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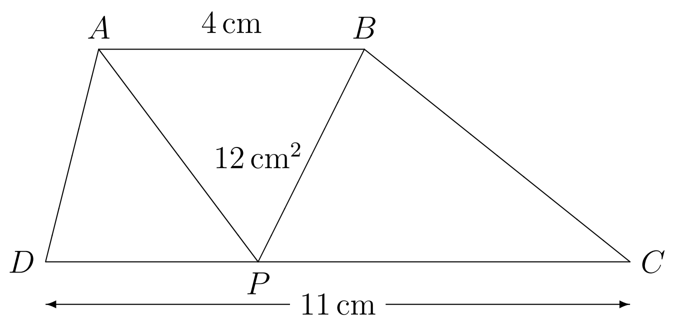
(1) $ABCD$ is a trapezium with AB parallel to DC, $AB = 4$ cm, $DC = 11$ cm and the area of triangle ABP is $12\,\text{cm}^2$. What is the area of the trapezium $ABCD$ in square centimetres?



[1 mark]

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(2) A library has 6 floors. There are 10 000 more books on the second floor than the first. The number of books on the third floor is the same as the number on the second. There are 10 000 fewer books on the fourth floor than the third and twice as many books on the fifth floor as there are on the fourth. On the sixth floor there are 4 000 fewer books than on the fifth. Coincidentally the number of books on the sixth floor is the same as the number on the first. Altogether, how many thousands of books are there in the library?

[1 mark]

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(3) XYZ is a three-digit number. Given that

$$XXXX + YYYY + ZZZZ = YXXXZ,$$

what is $X + Y + Z$?

[2 marks]

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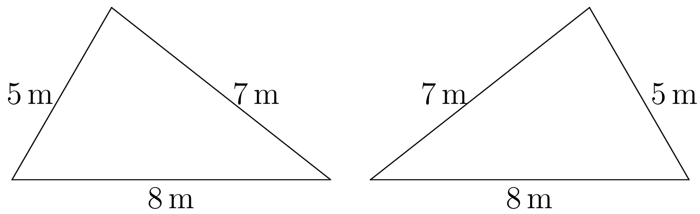
(4) Mathilda and her friends have 20 1-metre sticks out on the basketball court to make triangles. Below is a diagram of a triangle using all 20 sticks. It has sides of 5 m, 7 m and 8 m. All triangles with sides of 5 m, 7 m and 8 m (even mirror images as shown) are considered to be the same.



Altogether, how many different triangles could they make, *one at a time,* each time using all 20 sticks?

[2 marks]

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(5) What is the size of the angle, in degrees, between the hands of a clock when the time is ten past eleven?

[2 marks]

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(6) A plane is due to leave Perth at midnight and arrive at Tokyo at 12:29 pm, but it departs 49 minutes late. What percentage increase in speed is required in order that it will arrive exactly on time?

[3 marks]

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(7) Esther has 20 coins in her purse. They are 10c, 20c and 50c coins and the total value is \$5. If she has more 50c coins than 20c coins, how many 10c coins has she?

[3 marks]

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(8) Alice spent all her money in five shops. In each shop, she spent \$1 more than half of what she had when she entered that shop. How many dollars did Alice have when she entered the first shop?

[3 marks]

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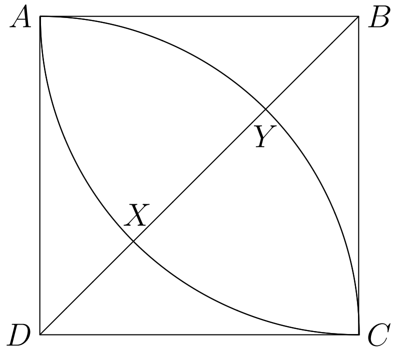
(9) Given a square $ABCD$, circular arcs centred at B and D are drawn from A to C. Now draw diagonal BD to cut these arcs at X and Y, respectively. If $XY = 12 - 6\sqrt{2}$, what is the area of the square $ABCD$?



[4 marks]

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(10) Farmer Brown runs a dairy farm with cows, sheep and goats. Dabbling in mathematics in his spare time, he noticed that the numbers of each animal were different prime numbers. He also observed that if he multiplied the number of cows by the total number of cows and sheep, he obtained a number just 120 greater than the number of goats. How many goats are there?

For full marks, explain how you found your solution.

[4 marks]

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Team Questions **45 minutes**

General instructions: *Calculators are (still)* ***not*** *permitted.*

A number appears on your computer screen. If you push the A key the number is decreased by 3, if you push the B key it's increased by 3 and if you press the C key the number is halved. You want to use a sequence of key strokes to change the number to 1.

For example, there are three ways to change 16 to 1:

- Use $CCCC$ giving the number sequence 16, 8, 4, 2, 1.
- Use $AAAAA$ giving 16, 13, 10, 7, 4, 1.
- Use CCA giving 16, 8, 4, 1.

So the shortest path from 16 to 1 requires 3 key strokes.

A. Find a starting number between 30 and 40 whose shortest path is 4, and show the path.

B. Find a starting number between 50 and 60 whose shortest path is 5, and show the path.

C. Find a starting number between 30 and 40 whose shortest path is 6, and show the path.

D. Which starting number between 40 and 50 has the longest "shortest path"?

E. Find a number which has two shortest paths, one beginning with the A key and one with the B key.

F. Find a starting number where the "add 3" operation must be used to get the shortest path.

G. What starting numbers can never get to 1?

H. What starting number between 500 and 1000 would have the shortest path?

I. In this question, the numbers on the screen don't have to be integers. If we start with any number then these two sequences of key strokes always produce the same final number: $ACCB$ and $BCBC$. Explain why.

Solutions

Solutions to Individual Questions

(1) Let h be the height of the trapezium. This is also the height of triangle ABP. Thus $12 = \frac{1}{2} \times 4h$ so $h = 6$. The area of the trapezium is then

$$\text{Area}_{\text{trap}} = h \times \frac{AB + DC}{2} = 6 \times \frac{4 + 11}{2} = 45.$$

(2) Let there be x thousand books on the first floor.
On the second floor there are $x + 10$ thousand books.
On the third floor there are $x + 10$ thousand books.
On the fourth floor there are x thousand books.
On the fifth floor there are $2x$ thousand books.
On the sixth floor there are $2x - 4$ thousand books.
Thus we have $2x - 4 = x$ and so $x = 4$. The total number of books, in thousands, is therefore

$$\begin{aligned} x + x + 10 + x + 10 + x + 2x + x &= 7x + 20 \\ &= 7 \times 4 + 20 = 48. \end{aligned}$$

(3) Writing the sum in the traditional way,

$$\begin{array}{r} XXXX \\ YYYY \\ ZZZZ \\ \hline YXXXZ \end{array}$$

we observe from the last column that $X + Y + Z = Z + 10k$, where k is the carry to the second column, and so $X + Y = 10k$. Now $X + Y$ can't be as big as 20, and, since XYZ is a 3-digit number, $X \neq 0$. Thus

$$X + Y = 10 \qquad (1)$$

and the carry $k = 1$. Looking at the second column now, we have $X + Y + Z + 1 = X + 10m$, i.e. $Y + Z + 1 = 10m$, where m is the carry to the third column. Again, $Y + Z + 1$ can't be as big as 20, so

$$Y + Z + 1 = 10 \qquad (2)$$

and $m = 1$. We see that this pattern is repeated for the third and fourth columns, and so the carry to the fifth column which is just Y, implies $Y = m = 1$. Adding equations (1) and (2) we have

$$\begin{aligned} X + Y + Y + Z + 1 &= 20 \\ X + Y + Z &= 20 - 1 - Y \\ &= 18 \end{aligned}$$

(Of course, we had enough information to evaluate that $X = 9$ and $Z = 8$.)

(4) We're looking for sets of three positive integers x, y, z satisfying $x+y+z=20$ and (so that we don't count the same set twice) $x \geq y \geq z$. With x the length of the hypotenuse, we also need $x > y+z$; otherwise we can't form a triangle. Listing systematically we get:

9,9,2
9,8,3
9,7,4
9,6,5
8,8,4
8,7,5
8,6,6
7,7,6.

So there are 8 possible triangles.

(5) The minute hand is 1/6 of a revolution past the 12, which is 60 degrees and the hour hand is 5/6 of the angle between 11 and 12 which is $5/6 = 1 - 1/6$ of 1/12 of a revolution before the 12, i.e. 25 degrees, so the angle between the hands is 85 degrees.

(6) The time usually taken for the trip is $12 \times 60 + 29 = 749$ minutes. In order to arrive on times the pilot must do the trip in $749 - 49 = 700$ minutes. Let the distance between Perth and Tokyo be d kilometres. The usual speed is therefore $d/749$, the required speed after the late departure is $d/700$. The required percentage increase in speed is therefore,

$$\begin{aligned}\frac{d/700 - d/749}{d/749} \times 100\% &= 749\Big(\frac{1}{700} - \frac{1}{749}\Big) \times 100\% \\ &= \Big(\frac{749}{700} - \frac{749}{749}\Big) \times 100\% \\ &= (1.07 - 1) \times 100\% \\ &= 7\%.\end{aligned}$$

So the required answer is 7.

(7) Let the number of 10c, 20c and 50c coins be x, y and z respectively. We have,

$$x + y + z = 20 \tag{3}$$

$$10x + 20y + 50z = 500 \tag{4}$$

and $z > y$. Dividing (4) by 10 and subtracting (3) gives

$$y + 4z = 30. \tag{5}$$

Since $z > y$ this means $5z > 30$ and so $z \geq 7$. If $z \geq 8$ the left hand side of (5) is at least 32, which is impossible, so $z = 7$ and we quickly find $y = 2$ and $x = 11$. So the number of ten cent coins is 11.

(8) In each shop she spends 2 more then she has when she leaves. So if she leaves with y dollars then she spent $y+2$ and entered with $2y+2$ dollars.
She leaves the last shop with \$0, so (using $y=0$), she entered shop 5 with \$2.
She left shop 4 with \$2, so entered it with $2 \times 2 + 2 = 6$ dollars.
She left shop 3 with \$6, so entered it with $2 \times 6 + 2 = 14$ dollars.
She left shop 2 with \$14, so entered with $2 \times 14 + 2 = 30$ dollars.
She left shop 1 with \$30, so entered with $2 \times 30 + 2 = 62$ dollars.
So Alice had 62 dollars when she entered the first shop.

(9) Let the side-length of the square be s. Now $BX = DY = s$, and, by Pythagoras' Theorem, $BD = s\sqrt{2}$. Therefore

$$XY = BX + DY - BD = (2 - \sqrt{2})s.$$

But we're told $XY = 12 - 6\sqrt{2} = 6(2 - \sqrt{2})$. Thus $s = 6$ and the square has area 36.

(10) Let the number of cows, sheep and goats be c, s and g respectively. The farmer notices that

$$c(c+s) = g + 120.$$

We know c, s and g are primes. If they were all odd then the left hand side of the above equation would be even and the right odd which is impossible. So one of them is even and therefore equals 2.
If $c = 2$ the left hand side of the equation is still even and the right hand side odd.
If $g = 2$ the equation becomes $c(c+s) = 122 = 2 \times 61$. This is impossible with c odd.
Therefore we must have $s = 2$ and get

$$\begin{aligned}
c^2 + 2c &= g + 120 \\
c^2 + 2c + 1 &= g + 121 \\
(c+1)^2 &= g + 11^2 \\
g &= (c+1)^2 - 11^2 \\
&= (c+1+11)(c+1-11)
\end{aligned}$$

Since g is a prime the first of the factors here must equal g and the other equal 1:

$$g = c + 1 + 11 \quad (6)$$
$$1 = c + 1 - 11 \quad (7)$$

From (7), we obtain $c = 11$; whence from (6) we deduce $g = 23$.

Solutions to Team Questions

(A) There is just one path with a starting number between 30 and 40 such that the shortest path from that starting number has length 4:

$$32, 16, 8, 4, 1$$

The sequence of operations giving this path is $CCCA$. An important observation here is that C effects a more rapid descent for numbers x that are even and greater than 4, but A requires just one step to convert 4 to 1 (compared with performing CC).

(B) There is just one path with a starting number between 50 and 60 such that the shortest path from that starting number has length 5:

$$56, 28, 14, 7, 4, 1$$

The sequence of operations giving this path is $CCCCA$.

(C) There are three paths with a starting number between 30 and 40 such that the shortest path from that starting number has length 6:

$$37, 34, 17, 14, 7, 4, 1 \quad \text{(by sequence: } ACACAA) \quad \text{or}$$
$$37, 40, 20, 10, 7, 4, 1 \quad \text{(by sequence: } BCCAAA) \quad \text{or}$$
$$37, 40, 20, 10, 5, 2, 1 \quad \text{(by sequence: } BCCCAC)$$

Thus all such paths start at 37. Thus we can see that a shortest path can involve the B operation.

(D) The starting number between 40 and 50 that has the longest "shortest path" is 49, with paths

$$49, 46, 23, 20, 10, 7, 4, 1 \quad (ACACAAA) \quad \text{or}$$
$$49, 46, 23, 20, 10, 5, 2, 1 \quad (ACACCAC) \quad \text{or}$$
$$49, 52, 26, 13, 10, 7, 4, 1 \quad (BCCAAAA) \quad \text{or}$$
$$49, 52, 26, 13, 10, 5, 2, 1 \quad (BCCACAC) \quad \text{or}$$
$$49, 52, 26, 13, 16, 8, 4, 1 \quad (BCCBCCA)$$

all of length 7. The path is not required.

(E) Starting numbers that have two shortest paths, one beginning with the A key and one with the B key, that are less than 100 are 13, 25, 37, 41, 49, 65, 73, 89, 97.

(F) Starting numbers where the "add 3" operation must be used to get the shortest path, that are less than 100 are 29, 53, 58, 61, 77, 85.

(G) The starting numbers that can never get to 1 are multiples of 3. (The way to see this is that if a number starts as a multiple of 3, then after any of the A, B or C operations is performed, the result is still a multiple of 3, and of course 1 is not a multiple of 3.)

(H) The starting number between 500 and 1000 that has the shortest path is 512.

(I) We need to show that $ACCB$ and $BCBC$ always produces the same result. Let x be the starting number. Then $ACCB$ produces

$$\begin{aligned}\big((x-3)/2\big)/2+3 &= \frac{x-3}{4}+3\\ &= \frac{x-3+12}{4} = \frac{x+9}{4}\end{aligned}$$

On the other hand, $BCBC$ produces

$$\begin{aligned}\big((x+3)/2+3\big)/2 &= \Big(\frac{x+3}{2}+3\Big)/2\\ &= \Big(\frac{x+3+6}{2}\Big)/2\\ &= \frac{x+9}{4}\end{aligned}$$

Thus irrespective of what starting number x is chosen. The result after each of the sequences $ACCB$ and $BCBC$ is $(x+9)/4$.