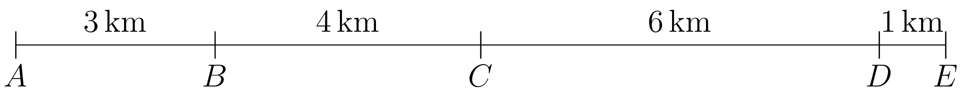
Western Australian Junior Mathematics Olympiad 2006

Individual Questions **100 minutes**

General instructions: *Each solution in this part is a positive integer less than 100. No working is needed for Questions 1 to 9. Calculators are **not** permitted. Write your answers on the answer sheet provided.*

---

1. The diagram below shows the train line from the outer suburb of A to the inner city station E, with the distance between stations shown in kilometres. After leaving a station the train travels at an average speed of 30 km/h for the first kilometre, then at 60 km/h until it reaches the next station. It spends 2 minutes at each station. How many minutes will elapse between the train leaving A and arriving at E?



[1 mark]

Solution. 60 km/h = 1 km/minute. So when the train travels at 60 km/h, it takes 1 minute to cover a km, and when it travels at 30 km/h, it takes 2 minutes to cover a km. So the minutes that will elapse between the train leaving A and arriving at E is given by:

$$(2+2)+2+(2+3)+2+(2+5)+2+(2+0)=24$$

where the first number (2) in each bracket is the time spent at 30 km/h, the second number in each bracket is the number of kilometres travelled at 60 km/h (which equals the time in minutes to cover that distance) and each unbracketed 2 is the waiting time at a station. Answer: 24.

---

2. Find the two-digit prime number that is 2 less than a perfect square that is 2 less than a prime. [1 mark]

Solution. Let the prime be p. Then p has 2 digits and hence is odd (2 is the only even prime). It is 2 less than a square. So the square is odd; possibilities are: 25, 49, 81. Of these only 81 has a prime both 2 less than it and 2 more than it. So p must be 79. Answer: 79.

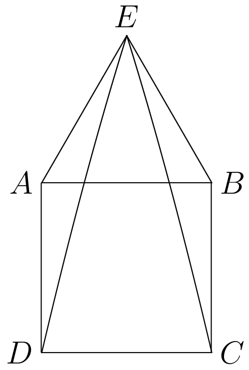
---

3. For how many positive integers (whole numbers) n are n, $3n$ and $n/3$ all three-digit integers? [2 marks]

Solution. The least that $n/3$ can be is 100 (the smallest 3-digit positive integer) and the most that $3n$ can be is 999 (the largest 3-digit positive integer). If $3n = 999$ then $n/3 = 111$. So $n/3$ can be any integer from 100 to 111 inclusive, a total of 12 possibilities, leading to 12 possibilities for n $(300, 303, \ldots, 333)$. Answer: 12.

---

4. A point E lies outside a square $ABCD$ so that ABE is an equilateral triangle. What is the measure in degrees of the angle CED?



[2 marks]

Solution. Since $\triangle ABE$ is equilateral, its angles are all $60°$. Hence $\angle EAD = 60° + 90°$. Now $\triangle ADE$ is isoceles. Thus

$$\begin{aligned}\angle AED &= \frac{1}{2}(180° - \angle EAD) \\ &= 15°\end{aligned}$$

Similarly, $\angle BEC = 15°$. So finally we have

$$\angle CED = 60° - 2 \times 15° = 30°.$$

Answer: 30.

---

5. A video store has a choice of 920 films to rent. You can rent some on DVD, some on video cassette and some on both. If the store owns a total of 1000 DVDs and video cassettes, how many films are available on both video cassette and DVD? [2 marks]

Solution. By the pigeon hole principle, 80 films are available on both video cassette and DVD (the films are the *pigeon holes;* put 920 of the 1000 DVDs and video cassettes in the 920 available pigeon holes; the remaining 80 DVDs or video cassettes must go in a pigeon hole that already contains a DVD or video cassette). Answer: 80.

---

6. Brenda was sick on the day of the maths test so she had to sit for it the next day. Her score of 96 raised the class average from 71 to 72. How many students (including Brenda) took the test?

[3 marks]

Solution. Let n be the number of students in the class. Without Brenda the average is 71 and hence the total marks without Brenda is $71(n-1)$. The average with Brenda's mark of 96 is 72. So

$$\begin{aligned} 72 &= \frac{71(n-1)+96}{n} \\ 72n &= 71(n-1)+96 = 71n - 71 + 96 \\ n &= 25 \end{aligned}$$

So there are 25 students in the class. Answer: 25.

---

7. A plane flies in still air at an average speed of 810 km/h for the duration of its flights. When flying from Perth to Sydney it takes four hours while from Sydney to Perth it takes five hours. Assuming that the wind is at a constant speed and from the west (i.e. in the direction from Perth to Sydney) for both flights, what is the speed of the wind?

[3 marks]

Solution. Let w be the speed of the wind (from west to east). Then travelling east the plane's speed is $(810+w)$ km/h and travelling west its speed is $(810-w)$ km/h. In general velocity v, distance d and time t are related by $v = d/t$. Let d now denote the distance from Perth to Sydney. Then

$$810 + w = \frac{d}{4} \tag{1}$$

$$810 - w = \frac{d}{5} \tag{2}$$

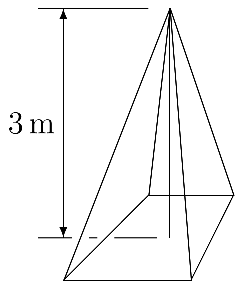
Eliminating d by dividing (1) by (2) we have

$$\begin{aligned} \frac{810+w}{810-w} &= \frac{5}{4} \\ 4(810+w) &= 5(810-w) \\ 4w &= 810 - 5w \\ 9w &= 810 \\ w &= 90 \end{aligned}$$

Therefore the wind speed w is 90 km/h. Answer: 90.

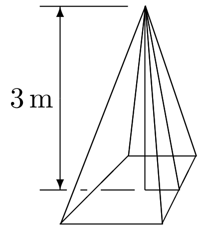
---

8. The five faces of a right square pyramid all have the same area. The height of the pyramid is 3 m. What is its total surface area in square metres?



[3 marks]

Solution.
Let a be the sidelength of the square base, so that the area of the base is a^2 which is also the area of each isosceles triangular face. So each triangular face has height $2a$, and the total surface area is $5a^2$. Using Pythagoras' Theorem and the pyramid height of 3 m, we have



$$\begin{aligned}(2a)^2 - (a/2)^2 &= 3^2\\ (4 - \tfrac{1}{4})a^2 = \tfrac{15}{4}a^2 = \tfrac{3}{4} \cdot 5a^2 &= 9\\ 5a^2 &= \tfrac{4}{3} \cdot 9 = 12\end{aligned}$$

So the total surface area of the pyramid is $12\,\text{m}^2$. Answer: 12.

---

9. Jasper glues together 504 cubes with 1 cm edges to make a solid rectangular-faced brick. If the perimeter of the base of the brick is 64 cm, what is the height of the brick? [4 marks]

Solution. $504 = 9 \times 56 = 9 \times 7 \times 8 = 2^3 \times 3^2 \times 7$. The volume of the brick $L \times B \times H = 504\,\text{cm}^3$, whereas the perimeter of the base $2(L + B) = 64\,\text{cm}$. So we need to write 504 as the product of three positive integers L, B, H such that $L + B = 32$. We see that $18 \times 14 \times 2 = 504$, so that (in cm) $L = 18$, $B = 14$, $H = 2$ is a solution.

To see the solution $H = 2$ is unique (not required for the competition), observe first that L must satisfy $16 \leq L < 32$ (taking $L \geq B$). Now L and B must both be even or both odd since there sum is even. If L and B are both odd then their product is 63 leading to $L = 21$ (to satisfy the inequality) and $B = 3$ which do not sum to 32. Hence L and B must both be even, whence $H = 1$ or 2. If $H = 1$ then the least $L + B$ could be is $2\sqrt{504} > 2 \times 22 = 44 > 32$. Thus $H = 2\,\text{cm}$. Answer: 2.

---

10. In the AFL Grand Final between the Knockers and the Beagles the Knockers won by 3 points. This was despite the Beagles kicking 16 scoring shots compared to the Knockers 14. It was also noticed that the number of behinds kicked by the Beagles was greater than the number of goals kicked by the Knockers, and the number of goals kicked by the Beagles was greater than the number of behinds kicked by the Knockers. How many points did the Beagles score?

Note: in the AFL game there are two ways to score: *behind*s score 1 point each, and *goal*s score 6 points each.

For full marks, explain how you found your solution. [4 marks]

Solution.

Let … B = the number of *goals* kicked by the *Beagles,*
b = the number of *behinds* kicked by the *Beagles,*
K = the number of *goals* kicked by the *Knockers,* and
k = the number of *behinds* kicked by the *Knockers.*

From the given information, we have:

$$B + b = 16 \quad (3)$$
$$K + k = 14 \quad (4)$$
$$6K + k = 6B + b + 3 \quad (5)$$
$$B > k \quad (6)$$
$$b > K \quad (7)$$

Since B and b are integers, from (6) and (7) we have

$$B \geq k + 1 \quad (8)$$
$$b \geq K + 1 \quad (9)$$

Adding the inequalities (8) and (9) and rearranging we get

$$(B + b) - (K + k) \geq 2.$$

But subtracting (3) and (4) gives

$$(B + b) - (K + k) = 2.$$

So the inequalities (8) and (9) must actually be equalities:

$$B = k + 1 \quad (10)$$
$$b = K + 1 \quad (11)$$

Substituting (10) in (3) and rearranging we have

$$B = 15 - K \quad (12)$$

Rearranging (4) we have

$$k = 14 - K \quad (13)$$

Substituting (11), (12) and (13) in (5) we have

$$\begin{aligned} 6K + 14 - K &= 6(15 - K) + K + 1 + 3 \\ 10K &= 80 \\ K &= 8 \end{aligned} \tag{14}$$

Substituting (14) in (12) and (11), we have:

$$B = 7 \tag{15}$$

$$b = 9 \tag{16}$$

so that the Beagles scored

$$6B + b = 51\,\text{points}$$

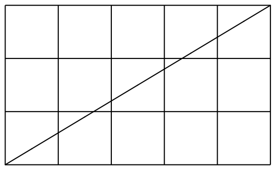
Note the problem can also be solved by systematically listing the possible scores for the Beagles (17 of them) and the possible scores for the Knockers (15 of them) and then doing a careful elimination ... and there are many other ways. The more satisfying solutions are ones that set up some inequality at the beginning that reduce the possibilities to a small list prior to doing an elimination.
Answer: 51.

Western Australian Junior Mathematics Olympiad 2006

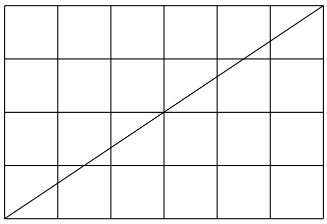
Team Questions — 45 minutes

General instructions: *Calculators are (still)* ***not*** *permitted.*

Consider a garden table made of 15 square tiles in a 5×3 arrangement.
The table has a straight crack along a diagonal.
Seven of the individual tiles are broken.



Now consider a 6×4 rectangle.
This time **eight** tiles are broken.



Try some other sizes. Use the squared paper provided.

A. How many tiles get broken when an 8×6 table is cracked along a diagonal?

Solution. 12 tiles get broken.

B. Give the dimensions of two different rectangular tables that get **nine** tiles broken when they are cracked along a diagonal.
Note. Remember that a square is also a rectangle. Also, note that for this and subsequent questions, an 8×6 table, for example, is considered the same as a 6×8 table.

Solution. The possible dimensions are: 9×1, 7×3, 9×3, 9×9.

C. How many different rectangular tables can you find that get **ten** tiles broken when they are cracked along a diagonal? Write down their dimensions.

Solution. The possible dimensions are: 10×1, 10×2, 10×5, 10×10, 9×2, 8×3, 7×4, 6×5.

D. Try some square tables. Describe what happens.

Solution. The number of broken tiles is the same as the length of the side of the square.

E. What happens when the shorter dimension of the table is 1?

Solution. Every tile is broken, i.e. the number of tiles is broken is the length of the rectangle.

F. For what sort of dimensions does the crack go through corners of tiles inside the rectangle?

Solution. For dimensions that have a common factor (larger than 1).

G. How many tiles are cracked when the diagonal does not go through any corner of a tile inside the rectangle? Explain your reasoning.

Solution. If the diagonal does not go through any corner of a tile inside the rectangle, a tile is cracked when and only when the diagonal enters a new column or enters a new row.

Say the table has m rows and n columns. The first tile broken is in the first row *and* the first column. Then there are $m-1$ further rows and $n-1$ further columns. Therefore the number of tiles cracked is $1+(m-1)+(n-1)=n+m-1$.

H. Predict the number of broken tiles in a 56×32 rectangle.

Solution. Remove the highest common factor 8. Then consider a 7×4 rectangle which has $7+4-1=10$ broken tiles. Reintroduce the removed common factor and the number of broken tiles is $10 \times 8 = 80$.

Alternatively: In general, the number of cracked tiles in a table with dimensions $m \times n$ is $m+n-\text{hcf}(m,n)$ where $\text{hcf}(m,n)$ counts the number of times the diagonal enters a new column and new row at the same time, which, if we think of the 'first' tile broken as being the bottom lefthand tile, is the number of times the crack goes through the bottom lefthand corner of a tile. So for a 56×32 rectangle, $56+32-\text{hcf}(56,32)=56+32-8=80$ tiles are broken.

I. Explain how you can predict the number of broken tiles in any size of table.

Solution. If the two dimensions have no common factor, add the two numbers and subtract 1. If the two dimensions do have a common factor, remove the highest common factor), consider the reduced table as above, then multiply the result by the removed HCF. Algebraically, the number of tiles broken is

$$\left(\frac{m}{\text{hcf}(m,n)}+\frac{n}{\text{hcf}(m,n)}-1\right)\text{hcf}(m,n)=m+n-\text{hcf}(m,n).$$

Alternatively: use the general argument in H to obtain the general formula $m+n-\text{hcf}(m,n)$.
